PHILOSOPHICAL TRANSACTIONS.

I. *On the Bodily Tides of Viscous and Semi-elastic Spheroids, and on the Ocean Tides upon a Yielding Nucleus.*

By G. H. DARWIN, *M.A., Fellow of Trinity College, Cambridge.*

Communicated by J. W. L. GLAISHER, *M.A., F.R.S.*

Received May 14,—Read May 23, 1878.

IN a well-known investigation Sir WILLIAM THOMSON has discussed the problem of the bodily tides of a homogeneous elastic sphere, and has drawn therefrom very important conclusions as to the great rigidity of the earth.*

Now it appears improbable that the earth should be perfectly elastic; for the contortions of geological strata show that the matter constituting the earth is somewhat plastic, at least near the surface. We know also that even the most refractory metals can be made to flow under the action of sufficiently great forces.

Although Sir W. THOMSON'S investigation has gone far to overthrow the old idea of a semi-fluid interior to the earth, yet geologists are so strongly impressed by the fact that enormous masses of rock are being, and have been, poured out of volcanic vents in the earth's surface, that the belief is not yet extinct that we live on a thin shell over a sea of molten lava. Under these circumstances it appears to be of interest to investigate the consequences which would arise from the supposition that the matter constituting the earth is of a viscous or imperfectly elastic nature; for if the interior *is* constituted in this way, then the solid crust, unless very thick, cannot possess rigidity enough to repress the tidal surgings, and these hypotheses must give results fairly conformable to the reality. The hypothesis of imperfect elasticity will be prin-

* Sir WILLIAM states that M. LAMÉ had treated the subject at an earlier date, but in an entirely different manner. I am not aware, however, that M. LAMÉ had fully discussed the subject in its physical aspect.

cipally interesting as showing how far Sir W. THOMSON'S results are modified by the supposition that the elasticity breaks down under continued stress.

In this paper, then, I follow out these hypotheses, and it will be seen that the results are fully as hostile to the idea of any great mobility of the interior of the earth as is that of Sir W. THOMSON.

The only terrestrial evidence of the existence of a bodily tide in the earth would be that the ocean tides would be less in height than is indicated by theory. The subject of this paper is therefore intimately connected with the theory of the ocean tides.

In the first part the equilibrium tide-theory is applied to estimate the reduction and alteration of phase of ocean tides as due to bodily tides, but that theory is acknowledged on all hands to be quite fallacious in its explanation of tides of short period.

In the second part of this paper, therefore, I have considered the dynamical theory of tides in an equatorial canal running round a tidally-distorted nucleus, and the results are almost the same as those given by the equilibrium theory.

The first two sections of the paper are occupied with the adaptation of Sir W. THOMSON'S work* to the present hypotheses; as, of course, it was impossible to reproduce the whole of his argument, I fear that the investigation will only be intelligible to those who are either already acquainted with that work, or who are willing to accept my quotations therefrom as established.

As some readers may like to know the results of this inquiry without going into the mathematics by which they are established, I have given in Part III. a summary of the whole, and have as far as possible relegated to that part of the paper the comments and conclusions to be drawn. I have tried, however, to give so much explanation in the body of the paper as will make it clear whither the argument is tending.

The case of pure viscosity is considered first, because the analysis is somewhat simpler, and because the results will afterwards admit of an easy extension to the case of elastico-viscosity.

I.

THE BODILY TIDES OF VISCOUS AND ELASTICO-VISCOUS SPHEROIDS.

1. *Analogy between the flow of a viscous body and the strain of an elastic one.*

The general equations of flow of a viscous fluid, *when the effects of inertia are neglected*, are

* His paper will be found in Phil. Trans., 1863, p. 573, and §§ 733–737 and 834–846 of THOMSON and TAIT'S 'Natural Philosophy,' edit. of 1867.

$$\left.\begin{aligned} -\frac{dp}{dx}+v\nabla^2\alpha+X=0 \\ -\frac{dp}{dy}+v\nabla^2\beta+Y=0 \\ -\frac{dp}{dz}+v\nabla^2\gamma+Z=0 \end{aligned}\right\} \quad \ldots \ldots \quad (1)$$

where x, y, z are the rectangular coordinates of a point of the fluid; α, β, γ are the component velocities parallel to the axes; p is the mean of the three pressures across planes perpendicular to the three axes respectively; X, Y, Z are the component forces acting on the fluid, estimated per unit volume; v is the coefficient of viscosity; and ∇^2 is the Laplacian operation $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$.

Besides these we have the equation of continuity $\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0$

Also if P, Q, R, S, T, U are the normal and tangential stresses estimated in the usual way across three planes perpendicular of the axes

$$\left.\begin{aligned} P=-p+2v\frac{d\alpha}{dx}, \quad Q=-p+2v\frac{d\beta}{dy}, \quad R=-p+2v\frac{d\gamma}{dz} \\ S=v\left(\frac{d\beta}{dz}+\frac{d\gamma}{dy}\right), \quad T=v\left(\frac{d\gamma}{dx}+\frac{d\alpha}{dz}\right), \quad U=v\left(\frac{d\alpha}{dy}+\frac{d\beta}{dx}\right) \end{aligned}\right\} \quad \ldots \quad (2)$$

Now in an elastic solid, if α, β, γ be the displacements, $m-\frac{1}{3}n$ be the coefficient of dilatation, and n that of rigidity, and if $\delta=\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}$; the equations of equilibrium are

$$\left.\begin{aligned} m\frac{d\delta}{dx}+n\nabla^2\alpha+X=0 \\ m\frac{d\delta}{dy}+n\nabla^2\beta+Y=0 \\ m\frac{d\delta}{dz}+n\nabla^2\gamma+Z=0 \end{aligned}\right\} \quad \ldots \ldots \quad (3)^*$$

Also

$$P=(m-n)\delta+2n\frac{d\alpha}{dx}, \quad Q=(m-n)\delta+2n\frac{d\beta}{dy}, \quad R=(m-n)\delta+2n\frac{d\gamma}{dz} \quad . \quad . \quad (4)$$

and S, T, U have the same forms as in (2), with n written instead of v.

Therefore if we put $-p=\frac{1}{3}(P+Q+R)$, we have $p=-(m-\frac{n}{3})\delta$, so that (3) may be written

* THOMSON and TAIT's 'Nat. Phil.,' § 698, eq. (7) and (8).

$$-\frac{m}{m-\frac{n}{3}}\frac{dp}{dx}+n\nabla^2\alpha+X=0,\ \&c.,\ \&c.$$

Also

$$P=-\frac{m-n}{m-\frac{n}{3}}p+2n\frac{da}{dx},\ Q=\ \&c.,\ R=\ \&c.$$

Now if we suppose the elastic solid to be incompressible, so that m is infinitely large compared to n, then it is clear that the equations of equilibrium of the incompressible elastic solid assume exactly the same form as those of flow of the viscous fluid, n merely taking the place of v.

Thus every problem in the equilibrium of an incompressible elastic solid has its counterpart in a problem touching the state of flow of an incompressible viscous fluid, when the effects of inertia are neglected; and the solution of the one may be made applicable to the other by merely reading for "displacements" "velocities," and for the coefficient of "rigidity" that of "viscosity."

2. *A sphere under influence of bodily force.*

Sir W. THOMSON has solved the following problem:—

To find the displacement of every point of the substance of an elastic sphere exposed to no surface traction, but deformed infinitesimally by an equilibrating system of forces acting *bodily* through the interior.

If for "displacement" we read velocity, and for "elastic" viscous, we have the corresponding problem with respect to a viscous fluid, and *mutatis mutandis* the solution is the same.

But we cannot find the tides of a viscous sphere by merely making the equilibrating system of forces equal to the tide-generating influence of the sun or moon, because the substance of the sphere must be supposed to have the power of gravitation.

For suppose that at any time the equation to the free surface of the earth (as the viscous sphere may be called for brevity) is $r=a+\sum_2^\infty\sigma_i$, where σ_i is a surface harmonic. Then the matter, positive or negative, filling the space represented by $\Sigma\sigma_i$ exercises an attraction on every point of the interior; and this attraction, together with that of a homogeneous sphere of radius a, must be added to the tide-generating influence to form the whole force in the interior of the sphere. Also it is a spheroid, and no longer a true sphere with which we have to deal. If, however, we cut a true sphere of radius a out of the spheroid (leaving out $\Sigma\sigma_i$), then by a proper choice of surface actions, the tidal problem may be reduced to finding the state of flow in a true sphere under the action of (i) an external tide-generating influence, (ii) the attraction of the true sphere, and of the positive and negative matter filling the space $\Sigma\sigma_i$, but (iii) subject to certain surface forces.

Since (i) and (ii) together constitute a bodily force, the problem only differs from that of Sir W. THOMSON in the fact that there are forces acting on the surface of the sphere.

Now as we are only going to consider small deviations from sphericity, these surface actions will be of small amount, and an approximation will be permissible.

It is clear that rigorously there is tangential action* between the layer of matter $\Sigma\sigma_i$ and the true sphere, but by far the larger part of the action is normal, and is simply the weight (either positive or negative) of the matter which lies above or below any point on the surface of the true sphere.

Thus, in order to reduce the earth to sphericity, the appropriate surface action is a normal traction equal to $-gw\Sigma\sigma_i$, where g is gravity at the surface, and w is the mass per unit volume of the matter constituting the earth.

In order to show what alteration this normal surface traction will make in Sir W. THOMSON's solution, I must now give a short account of his method of attacking the problem.

He first shows that, where there is a potential function, the solution of the problem may be subdivided, and that the complete values of α, β, γ consist of the sums of two parts which are to be found in different ways. The first part consists of *any* values of α, β, γ, which satisfy the equations throughout the sphere, without reference to surface conditions. As far as regards the second part, the bodily force is deemed to be non-existent and is replaced by certain surface actions, so calculated as to counteract the surface actions which correspond to the values of α, β, γ found in the first part of the solution. Thus the first part satisfies the condition that there is a bodily force, and the second adds the condition that the surface forces are zero. The first part of the solution is easily found, and for the second part Sir W. THOMSON discusses the case of an elastic sphere under the action of any surface tractions, but without any bodily force acting on it. The component surface tractions parallel to the three axes, in this problem, are supposed to be expanded in a series of surface harmonics; and the harmonic terms of any order are shown to have an effect on the displacements independent of those of every other order. Thus it is only necessary to consider the typical component surface tractions A_i, B_i, C_i of the order i.

He proves that (for an incompressible elastic solid for which m is infinite) this one surface traction A_i, B_i, C_i produces a displacement throughout the sphere given by

$$\alpha=\frac{1}{na^{i-1}}\left\{\frac{a^2-r^2}{2(2i^2+1)}\frac{d\Psi_{i-1}}{dx}+\frac{1}{i-1}\left[\frac{i+2}{(2i^2+1)(2i+1)}r^{2i+1}\frac{d}{dx}(\Psi_{i-1}r^{-2i+1})+\frac{1}{2i(2i+1)}\frac{d\Phi_{i+1}}{dx}+A_ir^i\right]\right\} \quad (5)\dagger$$

* I shall consider some of the effects of this tangential action in a future paper, viz.: "Problems connected with the Tides of a Viscous Spheroid," read before the Royal Society on December 19th, 1878.

† THOMSON and TAIT's 'Nat. Phil.,' 1867, § 737, equation (52).

with symmetrical expressions for β and γ; where Ψ and Φ are auxiliary functions defined by

$$\left.\begin{aligned}\Psi_{i-1}&=\frac{d}{dx}(A_i r^i)+\frac{d}{dy}(B_i r^i)+\frac{d}{dz}(C_i r^i)\\ \Phi_{i+1}&=r^{2i+3}\left\{\frac{d}{dx}(A_i r^{-i-1})+\frac{d}{dy}(B_i r^{-i-1})+\frac{d}{dz}(C_i r^{-i-1})\right\}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

In the case considered by Sir W. THOMSON of an elastic sphere deformed by bodily stress and subject to no surface action, we have to substitute in (5) and (6) only those surface actions which are equal and opposite to the surface forces corresponding to the first part of the solution;* but in the case which we now wish to consider, we must add to these latter the components of the normal traction $-gw\Sigma\sigma_i$, and besides must include in the bodily force both the external disturbing force, and the attraction of the matter of the spheroid on itself.

Now from the forms of (5) and (6) it is obvious that the tractions which correspond to the first part of the solution, and the traction $-gw\Sigma\sigma_i$ produce quite independent effects, and therefore we need only add to the complete solution of Sir W. THOMSON'S problem of the elastic sphere, the terms which arise from the normal traction $-gw\Sigma\sigma_i$. Finally we must pass from the elastic problem to the viscous one, by reading ν for n, and velocities for displacements.

I proceed then to find the state of internal flow in the viscous sphere, which results from a normal traction at every point of the surface of the sphere, given by the surface harmonic S_i.

In order to use the formulæ (5) and (6), it is first necessary to express the component tractions $\frac{x}{a}S_i$, $\frac{y}{a}S_i$, $\frac{z}{a}S_i$ as surface harmonics.

Now if V_i be a solid harmonic,

$$\frac{d}{dx}(r^{-2i-1}V_i)=-(2i+1)r^{-(2i+3)}xV_i+r^{-(2i+1)}\frac{dV_i}{dx}$$

So that

$$xV_i=\frac{1}{2i+1}\left\{r^2\frac{dV_i}{dx}-r^{2i+3}\frac{d}{dx}(r^{-2i-1}V_i)\right\}$$

Therefore

$$\frac{x}{a}S_i=\frac{1}{2i+1}\left\{\left[r^{-i+1}\frac{d}{dx}(r^iS_i)\right]-\left[r^{i+2}\frac{d}{dx}(r^{-i-1}S_i)\right]\right\}$$

The quantities within the brackets [ ] being independent of r, and being surface harmonics of orders $i-1$ and $i+1$ respectively, we have $\frac{x}{a}S_i$ expressed as the sum of two surface harmonics A_{i-1}, A_{i+1}, where

* Where the solid is incompressible, this surface traction is normal to the sphere at every point, provided that the potential of the bodily force is expressible in a series of solid harmonics.

$$A_{i-1}=\frac{1}{2i+1}r^{-i+1}\frac{d}{dx}(r^i S_i),\ A_{i+1}=-\frac{1}{2i+1}r^{i+2}\frac{d}{dx}(r^{-i-1}S_i)$$

Similarly $\frac{y}{a}S_i$, $\frac{z}{a}S_i$ may be expressed as $B_{i-1}+B_{i+1}$ and $C_{i-1}+C_{i+1}$, where the B's and C's only differ from the A's in having y, z written for x.

We have now to form the auxiliary functions Ψ_{i-1}, Φ_i corresponding to A_{i-1}, B_{i-1}, C_{i-1} and Ψ_i, Φ_{i+2} corresponding to A_{i+1}, B_{i+1}, C_{i+1}.

Then by the formulæ (6)

$$(2i+1)\Psi_{i-2}=\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)r^i S_i=0$$

$$\frac{2i+1}{r^{2i+1}}\Phi_i=\frac{d}{dx}\left[r^{-2i+1}\frac{d}{dx}(r^i S_i)\right]+\frac{d}{dy}\left[\quad\right]+\frac{d}{dz}\left[\quad\right]=-\frac{i(2i-1)}{r^{2i+1}}r^i S_i$$

$$-(2i+1)\Psi_i=\frac{d}{dx}\left[r^{2i+3}\frac{d}{dx}(r^{-i-1}S_i)\right]+\frac{d}{dy}\left[\quad\right]+\frac{d}{dz}\left[\quad\right]=-(i+1)(2i+3)r^i S_i$$

$$-\frac{2i+1}{r^{2i+5}}\Phi_{i+2}=\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)r^{-i-1}S_i=0$$

Thus

$$\Psi_{i-2}=0,\ \Phi_i=-\frac{i(2i-1)}{2i+1}r^i S_i,\ \Psi_i=\frac{(i+1)(2i+3)}{2i+1}r^i S_i,\ \Phi_{i+2}=0$$

Then by (5) we form α corresponding to A_{i-1}, B_{i-1}, C_{i-1}, and also to A_{i+1}, B_{i+1}, C_{i+1}, and add them together. The final result is that a normal traction S_i gives,

$$\alpha'=\frac{1}{va^i}\left[\left\{\frac{i(i+2)}{2(i-1)[2(i+1)^2+1]}a^2-\frac{(i+1)(2i+3)}{2(2i+1)[2(i+1)^2+1]}r^2\right\}\frac{d}{dx}(r^i S_i)\right.$$
$$\left.-\frac{i}{(2i+1)[2(i+1)^2+1]}r^{2i+3}\frac{d}{dx}(r^{-i-1}S_i)\right]\quad\ldots\ldots\quad(7)$$

and symmetrical expressions for β' and γ'.

α', β', γ' are here written for α, β, γ to show that this is only a partial solution, and v is written for n to show that it corresponds to the viscous problem. If we now put $S_i=-gw\sigma_i$, we get the state of flow of the fluid due to the transmitted pressure of the deficiencies and excesses of matter below and above the true spherical surface. This constitutes the solution as far as it depends on (iii).

There remain the parts dependent on (i) and (ii), which may for the present be classified together; and for this part Sir W. THOMSON's solution is directly applicable. The state of internal strain of an elastic sphere, subject to no surface action, but under the influence of a bodily force of which the potential is W_i, may be at once adapted to give the state of flow of a viscous sphere under like conditions. The solution is—

$$\alpha''=\frac{1}{v}\left[\left\{\frac{i(i+2)}{2(i-1)[2(i+1)^2+1]}a^2-\frac{(i+1)(2i+3)}{2(2i+1)[2(i+1)^2+1]}r^2\right\}\frac{dW_i}{dx}\right.$$
$$\left.-\frac{i}{(2i+1)[2(i+1)^2+1]}r^{2i+3}\frac{d}{dx}(r^{-2i-1}W_i)\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)^*$$

with symmetrical expressions for β'' and γ''.

I will first consider (ii); *i.e.*, the matter of the earth is now supposed to possess the power of gravitation.

The gravitation potential of the spheroid $r=a+\sigma_i$ (taking only a typical term of σ) at a point in the interior, estimated per unit volume, is

$$\frac{gw}{2a}(3a^2-r^2)+\frac{3gw}{2i+1}\left(\frac{r}{a}\right)^i\sigma_i$$

according to the usual formula in the theory of the potential.

Now the first term, being symmetrical round the centre of the sphere, can clearly cause no flow in the incompressible viscous sphere. We are therefore left with $\frac{3gw}{2i+1}\left(\frac{r}{a}\right)^i\sigma_i$.

Now if $\frac{3gw}{2i+1}\left(\frac{r}{a}\right)^i\sigma_i$ be substituted for W_i in (8), and if the resulting expression be compared with (7) when $-gw\sigma_i$ is written for S_i, it will be seen that $-\alpha''=\frac{3}{2i+1}\alpha'$.

Thus

$$\alpha'+\alpha''=\alpha''\left(1-\frac{2i+1}{3}\right)^{\dagger}=-\frac{2}{3}(i-1)\alpha''.$$

And if $V_i=\frac{3gw}{2i+1}\left(\frac{r}{a}\right)^i\sigma_i$,

$$\alpha'+\alpha''=-\frac{1}{v}\left[\left\{\frac{i(i+2)}{2(i-1)[2(i+1)^2+1]}a^2-\frac{(i+1)(2i+3)}{2(2i+1)[2(i+1)^2+1]}r^2\right\}\frac{d}{dx}\left(\frac{2}{3}(i-1)V_i\right)\right.$$
$$\left.-\frac{i}{(2i+1)[2(i+1)^2+1]}r^{2i+3}\frac{d}{dx}\left(r^{-2i-1}\frac{2}{3}(i-1)V_i\right)\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

with symmetrical expressions for $\beta'+\beta''$ and $\gamma'+\gamma''$.

Equation (9) then embodies the solution as far as it depends on (ii) and (iii). And since (9) is the same as (8) when $-\frac{2}{3}(i-1)V_i$ is written for W_i, we may include all the effects of mutual gravitation in producing a state of flow in the viscous sphere, by adopting THOMSON's solution (8), and taking instead of the true potential of the layer

* 'Nat. Phil.', § 834, equation (8) when m is infinite compared with n, and $i-1$ written for i, and v replaces n.

† The case of § 815 in THOMSON and TAIT's 'Nat. Phil.' is a special case of this.

of matter σ_i, $-\frac{2}{3}(i-1)$ times that potential, and by adding to it the external disturbing potential.

We have now learnt how to include the surface action in the potential; and if W_i be the potential of the external disturbing influence, the *effective* potential per unit volume at a point within the sphere, now free of surface action and of mutual gravitation, is $W_i-\frac{2gw(i-1)}{2i+1}\left(\frac{r}{a}\right)^i\sigma_i=r^iT_i$ suppose.

The complete solution of our problem is then found by writing r^iT_i in place of W_i in THOMSON's solution (8).*

In order however to apply the solution to the case of the earth, it will be convenient to use polar coordinates. For this purpose, write wr^iS_i for W_i, and let r be the radius vector; θ the colatitude; ϕ the longitude. Let ρ, ϖ, ν be the velocities radially, and along and perpendicular to the meridian respectively. Then the expressions for ρ, ϖ, ν will be precisely the same as those for α, β, γ in (8), save that for $\frac{d}{dx}$ we must put $\frac{d}{dr}$; for $\frac{d}{dy}$, $\frac{d}{r\sin\theta d\phi}$; and for $\frac{d}{dz}$, $\frac{d}{rd\theta}$.

Then after some reductions we have

$$\left.\begin{aligned}\rho&=\frac{i^2(i+2)a^2-i(i^2-1)r^2}{2(i-1)[2(i+1)^2+1]\upsilon}r^{i-1}T_i\\ \varpi&=\frac{i(i+2)a^2-(i-1)(i+3)r^2}{2(i-1)[2(i+1)^2+1]\upsilon}r^{i-1}\frac{dT_i}{d\theta}\\ \nu&=\frac{i(i+2)a^2-(i-1)(i+3)r^2}{2(i-1)[2(i+1)^2+1]\upsilon}\frac{r^{i-1}}{\sin\theta}\frac{dT_i}{d\phi}\end{aligned}\right\}\quad\dots\dots\quad(10)\dagger$$

where $T_i=w\left(S_i-2g\frac{i-1}{2i+1}\frac{\sigma_i}{a^i}\right)$.

These equations for ρ, ϖ, ν give us the state of internal flow corresponding to the external disturbing potential r^iS_i, including the effects of the mutual gravitation of the matter constituting the spheroid.

* The introduction of the effects of gravitation may be also carried out synthetically, as is done by Sir W. THOMSON (§ 840, 'Nat. Phil.'); but the effects of the lagging of the tide-wave render this method somewhat artificial, and I prefer to exhibit the proof in the manner here given. Conversely, the elastic problem may be solved as in the text.

† There seems to be a misprint as to the signs of the Θ's in the second and third of equations (13) of § 834 of the 'Nat. Phil.' (1867). When this is corrected μ and ν admit of reduction to tolerably simple forms. It appears to me also that the differentiation of ρ in (15) is incorrect; and this falsifies the argument in three following lines. The correction is not, however, in any way important.

3. *The form of the free surface at any time.*

If ρ' be the surface value of ρ, then

$$\rho'=\frac{i(2i+1)}{2(i-1)[2(i+1)^2+1]}\frac{a^{i+1}}{v}\mathrm{T}_i$$

Hence after a short interval of time δt, the equation to the bounding surface of the spheroid becomes $r=a+\sigma_i+\rho'\delta t$; but during this same interval, σ_i has become $\frac{d\sigma_i}{dt}\delta t$, whence

$$\frac{d\sigma_i}{dt}=\rho'=\frac{i(2i+1)}{2(i-1)[2(i+1)^2+1]}\frac{wa^{i+1}}{v}\mathrm{S}_i-\frac{i}{2(i+1)^2+1}\frac{gwa}{v}\sigma_i$$

or

$$\frac{d\sigma_i}{dt}+\frac{i}{2(i+1)^2+1}\frac{gwa}{v}\sigma_i=\frac{i(2i+1)}{2(i-1)[2(i+1)^2+1]}\frac{wa^{i+1}}{v}\mathrm{S}_i \quad . \quad . \quad . \quad . \quad (11)$$

This differential equation gives the manner in which the surface changes, under the influence of the external potential $r^i\mathrm{S}_i$.

If S_i be not a function of the time, and if s_i be the value of σ_i when $t=0$,

$$\sigma_i=\frac{2i+1}{2(i-1)}\frac{a^i\mathrm{S}_i}{g}\left[1-exp\left(\frac{-gwait}{[2(i+1)^2+1]v}\right)\right]+s_i\, exp\left(\frac{-gwait}{[2(i+1)^2+1]v}\right) \quad . \quad . \quad (12)^*$$

When t is infinite

$$\sigma_i=\frac{2i+1}{2(i-1)}\frac{a^i\mathrm{S}_i}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and there is no further state of flow, for the fluid has assumed the form which it would have done if it had not been viscous. This result is of course in accordance with the equilibrium theory of tides.

If S_i be zero, the equation shows how the inequalities on the surface of a viscous globe would gradually subside under the influence of simple gravity. We see how much more slowly the change takes place if i be large; that is to say, inequalities of small extent die out much more slowly than wide-spread inequalities. Is it not possible that this solution may throw some light on the laws of geological subsidence and upheaval?

4. *Digression on the adjustments of the earth to a form of equilibrium.*

In a former paper I had occasion to refer to some points touching the precession of a viscous spheroid, and to consider its rate of adjustment to a new form of equilibrium,

* I write "exp." for "e to the power of."

when its axis of rotation had come to depart from its axis of symmetry.* I propose then to discuss the subject shortly, and to establish the law which was there assumed.

Suppose that the earth is rotating with an angular velocity ω about the axis of z, but that at the instant at which we commence our consideration the axis of symmetry is inclined to the axis of z at an angle α in the plane of xy, and that at that instant the equation to the free surface is

$$r=a\left\{1+\frac{5m}{4}\left(\frac{1}{3}-[\cos\alpha\cos\theta+\sin\alpha\sin\theta\cos\phi]^2\right)\right\}$$

where m is the ratio of centrifugal force at the equator to pure gravity, and therefore equal to $\frac{\omega^2 a}{g}$.

Then putting $i=2$ in (12), and dropping the suffixes of S, s, σ, $s=\frac{5ma}{4}\left(\frac{1}{3}-[\quad]^2\right)$.

We may conceive the earth to be at rest, if we apply a potential

$$wr^2\mathrm{S}=\frac{1}{2}\omega^2 wr^2\left(\frac{1}{3}-\cos^2\theta\right)$$

so that

$$\mathrm{S}=\frac{1}{2}\omega^2\left(\frac{1}{3}-\cos^2\theta\right)$$

By (12) we have

$$\sigma=\frac{5a^2\mathrm{S}}{2g}\left[1-exp\left(-\frac{2wgat}{19v}\right)\right]+s\ exp\left(-\frac{2wgat}{19v}\right)$$

Then, substituting for S and s, and putting $\kappa=\frac{2wga}{19v}$

$$\sigma=\frac{5ma}{4}\left\{\left(\frac{1}{3}-\cos^2\theta\right)[1-exp(-\kappa t)]+\left(\frac{1}{3}-[\cos\alpha\cos\theta+\sin\alpha\sin\theta\cos\phi]^2\right)exp(-\kappa t)\right\}$$

Now

$$\begin{aligned}&[1-exp(-\kappa t)]\cos^2\theta+exp(-\kappa t)(\cos\alpha\cos\theta+\sin\alpha\sin\theta\cos\phi)^2\\&=\cos^2\theta\,[1-\sin^2\alpha\,exp(-\kappa t)]+\sin^2\alpha\sin^2\theta\cos^2\phi\,exp(-\kappa t)\\&+2\sin\alpha\cos\alpha\sin\theta\cos\theta\cos\phi\,exp(-\kappa t).\end{aligned}$$

Therefore the Cartesian equation to the spheroid at the time t is,

$$\frac{x^2+y^2+z^2}{1+\frac{5m}{6}}=a^2-\frac{5m}{2}\{z^2(1-\sin^2\alpha\,exp(-\kappa t))+x^2\sin^2\alpha\,exp(-\kappa t)+2xz\sin\alpha\cos\alpha\,exp(-\kappa t)\}$$

or

* "On the Influence of Geological Changes on the Earth's Axis of Rotation," Phil. Trans., Vol. 167, Part I., sec. 5.

$$x^2\left\{1+\frac{5m}{2}\sin^2\alpha\, exp(-\kappa t)\right\}+y^2+z^2\left\{1+\frac{5m}{2}(1-\sin^2\alpha\, exp(-\kappa t))\right\}$$
$$+5m\sin\alpha\cos\alpha\, xz\, exp(-\kappa t)=a^2\left(1+\frac{5m}{6}\right)$$

Let α' be the inclination of the principal axis at this time to the axis of z, then

$$\tan 2\alpha'=\frac{\sin 2\alpha\, exp(-\kappa t)}{1-2\sin^2\alpha\, exp(-\kappa t)}$$

If α be small, as it was in the case I considered in my former paper, then $\alpha'=\alpha\, exp(-\kappa t)$ and $\frac{d\alpha'}{dt}=-\kappa\alpha'$.

Therefore the velocity of approach of the principal axis to the axis of rotation varies as the angle between them, which is the law assumed.

Also $\kappa=\frac{2wga}{19v}$, so that κ (the ν of my former paper) varies inversely as the coefficient of viscosity,—as was also assumed.

5. *Bodily tides in a viscous earth.**

The only case of interest in which S_i of equation (11) is a function of the time, is where it is a surface harmonic of the second order, and is periodic in time; for this will give the solution of the tidal problem. Since, moreover, we are only interested in the case where the motion has attained a permanently periodic character, the exponential terms in the solution of (11) may be set aside.

Let $S_2=S\cos(vt+\eta)$, and in accordance with THOMSON's notation,† let $\frac{2g}{5a}=\mathfrak{g}$, and $\frac{19v}{5wa^2}=\mathfrak{r}$; and therefore $\frac{2gwa}{19v}=\frac{\mathfrak{g}}{\mathfrak{r}}$.

Then putting $i=2$ in (11), and omitting the suffix of σ for brevity, we have

$$\frac{d\sigma}{dt}+\frac{\mathfrak{g}}{\mathfrak{r}}\sigma=\frac{a}{\mathfrak{r}}S\cos(vt+\eta)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

It is evident that σ must be of the form $A\cos(vt+B)$, and therefore

$$A\{-v\mathfrak{r}\sin(vt+B)+\mathfrak{g}\cos(vt+B)\}=aS\cos(vt+\eta)$$

* In certain cases the forces do not form a rigorously equilibrating system, but there is a very small couple tending to turn the earth. The effects of this unbalanced couple, which varies as the square of $\frac{3}{2}\frac{m}{c^3}$, will be considered in a succeeding paper on the "Precession of a Viscous Spheroid." (Read before the Royal Society, December 19th, 1878.)

† 'Nat. Phil.', § 840, eq. (27).

or if we put $\tan\epsilon=\frac{v\mathfrak{r}}{\mathfrak{g}}$,

$$A\mathfrak{g}\sec\epsilon\cos(vt+B+\epsilon)=aS\cos(vt+\eta).$$

Hence $A=\frac{a}{\mathfrak{g}}S\cos\epsilon$, and $B=\eta-\epsilon$.

Therefore the solution of (14) is,

$$\sigma=\frac{a}{\mathfrak{g}}S\cos\epsilon\cos(vt+\eta-\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

Where $\tan\epsilon=\frac{v\mathfrak{r}}{\mathfrak{g}}=\frac{19vv}{2gaw}$

But if the globe were a perfect fluid, and if the equilibrium theory of tides were true, we should have by (13),

$$\sigma=\frac{5a}{2g}\cdot aS\cos(vt+\eta)=\frac{Sa}{\mathfrak{g}}\cos(vt+\eta).$$

Thus we see that the tides of the viscous sphere are to the equilibrium tides of a fluid sphere as $\cos\epsilon : 1$, and that there is a retardation in time of $\frac{\epsilon}{v}$.

A parallel investigation will be applicable to the general case where the disturbing potential is $wr^i S_i\cos(vt+\eta)$; and the same solution will be found to hold save that we now have $\tan\epsilon=\frac{2(i+1)^2+1}{i}\cdot\frac{vv}{gaw}$, and that in place of $\mathfrak{g}$ we have $\frac{2(i-1)g}{(2i+1)a}$.

6. *Diminution of ocean tides on equilibrium theory.*

Suppose now that there is a shallow ocean on the viscous nucleus, and let us find the effects on the ocean tides of the motion of the nucleus according to the equilibrium theory, neglecting the gravitation of the water.

The potential at a point outside the nucleus is

$$g\frac{a^2}{r}+\frac{3}{5}g\left(\frac{a}{r}\right)^2\sigma+r^2S\cos(vt+\eta),$$

and if this be put equal to a constant, we get the form which the ocean must assume. Let $r=a+u$ be the equation to the surface of the ocean. Then substituting for r in the potential, and neglecting u in the small terms, and equating the whole to a constant, we find,

$$-gu+\frac{3}{5}g\sigma+a^2S\cos(vt+\eta)=0$$

or

$$u=\frac{3}{5}\sigma+\frac{a^2}{g}S\cos(vt+\eta).$$

But the rise and fall of the tide relative to the nucleus is given by $u-\sigma$, and

$$u-\sigma=\frac{a^2\mathrm{S}}{g}\cos(vt+\eta)-\frac{2}{5}\sigma$$

$$=\frac{2}{5}\frac{a\mathrm{S}}{\mathfrak{g}}[\cos(vt+\eta)-\cos\epsilon\cos(vt+\eta-\epsilon)]$$

$$=-\frac{2}{5}\frac{a\mathrm{S}}{\mathfrak{g}}\sin\epsilon\sin(vt+\eta-\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Now if the nucleus had been rigid, the rise and fall would have been given by

$$\frac{2}{5}\frac{a\mathrm{S}}{\mathfrak{g}}\cos(vt+\eta)=\mathrm{H}\cos(vt+\eta) \text{ suppose.}$$

Therefore

$$u-\sigma=-\mathrm{H}\sin\epsilon\sin(vt+\eta-\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Hence the apparent tides on the yielding nucleus are equal to the tides on a rigid nucleus reduced in the proportion $\sin\epsilon : 1$; and since $-\sin(vt+\eta-\epsilon) = \cos\left(vt+\eta+\frac{\pi}{2}-\epsilon\right)$ they are retarded by $\frac{1}{v}\left(\epsilon-\frac{\pi}{2}\right)$. As ϵ is necessarily less than $\frac{\pi}{2}$, this is equivalent to an acceleration of the time of high water equal to $\frac{1}{v}\left(\frac{\pi}{2}-\epsilon\right)$.

It is, however, worthy of notice that this is only an acceleration of phase relatively to the nucleus, and there is an absolute retardation of phase equal to arc-tan $\frac{3\sin\epsilon\cos\epsilon}{3+2\cos^2\epsilon}$.

7. *Semidiurnal and fortnightly tides.*

Let the axis of z be the earth's axis of rotation, and let the plane of xz be fixed in the earth; let c be the moon's distance, and m its mass.

Suppose the moon to move in the equator with an angular velocity ω relatively to the earth, and let the moon's terrestrial longitude, measured from the plane of xz, at the time t be ωt.

Then at the time t, the gravitation potential of the tide generating force, estimated per unit volume of the earth's mass is

$$-\frac{3}{2}\frac{m}{c^3}wr^2\left\{\frac{1}{3}-\sin^2\theta\cos^2(\phi-\omega t)\right\}$$

which is equal to

$$\frac{3}{4}\frac{m}{c^3}wr^2\left(\frac{1}{3}-\cos^2\theta\right)+\frac{3}{4}\frac{m}{c^3}wr^2\{\sin^2\theta\cos 2\phi\cos 2\omega t+\sin^2\theta\sin 2\phi\sin 2\omega t\}.$$

The first term of this expression is independent of the time, and therefore produces an effect on the viscous earth, which will have died out when the motion has become steady; its only effect is slightly to increase the ellipticity of the earth's surface.

The two latter terms give rise to two tides, in one of which (according to previous notation)

$$S \cos(vt+\eta)=\frac{3}{4}\frac{m}{c^3}\sin^2\theta\cos 2\phi\cos 2\omega t,$$

and in the second of which

$$S \cos(vt+\eta)=-\frac{3}{4}\frac{m}{c^3}\sin^2\theta\sin 2\phi\cos\left(2\omega t+\frac{\pi}{2}\right).$$

Now ϵ, which depends on the frequency of the tide generating potential, will clearly be the same for both these tides; and therefore they will each be equal to the corresponding tides of a fluid spheroid, reduced by the same amount and subject to the same retardation. They may therefore be recompounded into a single tide; and since v will here be equal to 2ω, it follows that the retardation of the bodily semi-diurnal tide is $\frac{\epsilon}{2\omega}$, where $\tan\epsilon=\frac{2\omega\mathfrak{r}}{\mathfrak{g}}=\frac{19 \upsilon\omega}{gaw}$. Also the height of the tide is less than the corresponding equilibrium tide of a fluid spheroid in the proportion of $\cos\epsilon$ to unity.

Similarly by section (6) the height of the ocean tide on the yielding nucleus is given by the corresponding tide on a rigid nucleus multiplied by $\sin\epsilon$, and there is an acceleration of relative high water equal to $\frac{\pi}{4\omega}-\frac{\epsilon}{2\omega}$.

The case of the fortnightly tide is somewhat simpler.

If Ω be the moon's orbital angular velocity, and I the inclination of the plane of the orbit to the earth's equator, then the part of the tide generating potential, on which the fortnightly tide depends, is—

$$\frac{9}{8}\frac{m}{c^3}wr^2\sin^2 I\left(\tfrac{1}{3}-\cos^2\theta\right)\cos 2\Omega t$$

and we see at once by sections (5) and (6) that $\tan\epsilon=\frac{19\,\upsilon\Omega}{gaw}$. The bodily tide is the tide of a fluid spheroid multiplied by $\cos\epsilon$; the reduction of ocean tide is given by $\sin\epsilon$; and there is a time-acceleration of relative high water of $\frac{\pi}{4\Omega}-\frac{\epsilon}{2\Omega}$ or $\frac{1}{2}-\frac{\epsilon}{\pi}$ of a week.

In order to make the meaning of the previous analytical results clearer, I have formed the following numerical tables, to show the effects of this hypothesis on the semidiurnal and fortnightly tides. The coefficient of viscosity is usually expressed in gravitation units of force so that the formula for ϵ becomes, $\tan\epsilon=\frac{19\,\upsilon\omega}{wa}$. In the tables υ is expressed in the centimetre-gramme-second system, and in gravitation units of force; a is taken as $6{\cdot}37\times10^8$, and w as $5{\cdot}5$, and the angular velocity ω of the moon relatively to the earth as ·00007025 radians per second.

With these data I find $\upsilon=10^{12}\times 2{\cdot}625\tan\epsilon$. As a standard of comparison with the coefficients of viscosity given in the tables, I may mention that, according to some rough experiments of my own, the viscosity of British pitch at near the freezing temperature (34° Fahr.), when it is hard and brittle, is about $10^8 \times 1 \cdot 3$ when measured in the same units.

Lunar Semidiurnal Tide.

Coefficient of viscosity $\times 10^{-10}$ $(\nu \times 10^{-10})$	Retardation of bodily tide $\left(\frac{\epsilon}{2\omega}\right)$		Height of bodily tide is tide of fluid spheroid multiplied by $(\cos \epsilon)$.	Height of ocean tide is tide on rigid nucleus multiplied by $(\sin \epsilon)$.	High tide relatively to viscous nucleus accelerated by $\frac{1}{\omega}\left(\frac{\pi}{4} - \frac{\epsilon}{2}\right)$	
	Hrs.	min.			Hrs.	min.
Fluid 0	0	0	1·000	·000	3	6
46	0	21	·985	·174	2	46
96	0	41	·940	·342	2	25
152	1	2	·866	·500	2	4
220	1	23	·766	·643	1	44
313	1	44	·643	·766	1	23
455	2	4	·500	·866	1	2
721	2	25	·342	·940	0	41
1,488	2	46	·174	·985	0	21
Rigid ∞	3	6	·000	1·000	0	0

Fortnightly Tide.

Coefficient of viscosity $\times 10^{-10}$	Retardation		Height of bodily tide	Height of ocean tide	High tide accelerated by	
	Days.	hrs.			Days.	hrs.
Fluid 0	0	0	1·000	·000	3	10
1,200	0	9	·985	·174	3	1
2,500	0	18	·940	·342	2	16
4,000	1	3	·866	·500	2	6
5,800	1	12	·766	·643	1	21
8,300	1	21	·643	·766	1	12
12,000	2	6	·500	·866	1	3
19,000	2	16	·342	·940	0	18
39,300	3	1	·174	·985	0	9
Rigid ∞	3	10	·000	1·000	0	0

I now pass on to a case which is intermediate between the hypothesis of Sir W. THOMSON and that just treated.

8. *The tides of an elastico-viscous spheroid.*

The term elastico-viscous is used to denote that the stresses requisite to maintain the body in a given strained configuration decrease the longer the body is thus constrained, and this is undoubtedly the case with many solids. In the particular case which is here treated, it is assumed that the stresses diminish in geometrical progression, as the time increases in arithmetical progression. If, for example, a cubical block of the substance be strained to a given amount by a shearing stress T, and maintained in that position, then after a time t, the shearing stress, is $\mathrm{T}\,exp\left(-\frac{t}{\mathfrak{t}}\right)$.

The time $\mathbf{t}$ measures the rate at which the stress falls off, and is called (I believe by Professor MAXWELL) "the modulus of the time of relaxation of rigidity;" it is the time in which the initial stress has been reduced to e^{-1} or ·3679 of its initial value. I do not suppose, however, that any solid conforms exactly to this law; but I conceive that it is often useful in physical problems to discuss mathematically an ideal case, which presents a sufficiently marked likeness to the reality, where we are unable to determine exactly what that reality may be.

Mr. J. G. BUTCHER has found the equations of motion of such an ideal substance from the consideration that the elasticity of groups of molecules is continually breaking down, and that the groups rearrange themselves afterwards.* These considerations lead him to the following results for the stresses across rectangular planes at any point in the interior, viz. (with the notation of § 1):—

$$\mathrm{P}=(m-n)\delta+2n\left(\frac{1}{\mathbf{t}}+\frac{d}{dt}\right)^{-1}\left(\frac{d\alpha}{dx}+\frac{1}{3\mathbf{t}}\alpha\right),\ \mathrm{S}=n\left(\frac{1}{\mathbf{t}}+\frac{d}{dt}\right)^{-1}\left(\frac{d\beta}{dz}+\frac{d\gamma}{dy}\right)$$

and similar expressions for Q, R, T, U; where $m-\frac{n}{3}$ is the coefficient of dilatation, n that of rigidity, δ the dilatation, and α, β, γ, the components of flow.

These expressions are clearly in accordance with the above definition of elastico-viscosity, for $\frac{d\mathrm{S}}{dt}+\frac{\mathrm{S}}{\mathbf{t}}=n\left(\frac{d\beta}{dz}+\frac{d\gamma}{dy}\right)$.

If the expressions for P, S, &c., be substituted in the equations of equilibrium of the elementary parallelopiped, it is found by aid of the equation of continuity $\frac{d\delta}{dt}=\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}$, that when inertia is neglected

$$\left(\frac{1}{\mathbf{t}}+\frac{d}{dt}\right)^{-1}\left\{\left[\left(m-\tfrac{1}{3}n\right)\frac{1}{\mathbf{t}}+m\frac{d}{dt}\right]\frac{d\delta}{dx}+n\nabla^2\alpha\right\}+\mathrm{X}=0$$

and two similar equations.

By the same reasoning as in § 1, we may put, $\delta=\frac{-p}{m-\frac{1}{3}n}$, and the equations become

$$-\left(\frac{1}{\mathbf{t}}+\frac{d}{dt}\right)^{-1}\left\{\left[\frac{1}{\mathbf{t}}+\frac{m}{m-\frac{1}{3}n}\frac{d}{dt}\right]\frac{dp}{dx}-n\nabla^2\alpha\right\}+\mathrm{X}=0.$$

Then supposing the substance to be incompressible, so that m is infinitely large compared to n, and therefore $m\div m-\frac{1}{3}n$ is unity, the equations become

$$-\frac{dp}{dx}+n\left(\frac{1}{\mathbf{t}}+\frac{d}{dt}\right)^{-1}\nabla^2\alpha+\mathrm{X}=0$$

and two similar equations.

* Proc. Lond. Math. Soc., Dec. 14, 1876, p. 107–9. It seems to me that the hypothesis ought to represent the elastico-viscosity of ice very closely.

Now these equations have exactly the same form as those for the motion of a viscous fluid, save that the coefficient of viscosity v is replaced by $n\left(\frac{1}{\mathfrak{t}}+\frac{d}{dt}\right)^{-1}$. We may therefore at once pass to the differential equation (11) which gives the form of the surface of the spheroid at any time.

Substituting, therefore, in (11) for $\frac{1}{\nu}, \frac{1}{n}\left(\frac{1}{\mathfrak{t}}+\frac{d}{dt}\right)$, we get

$$\left[1+\frac{i}{2(i+1)^2+1}\frac{gwa}{n}\right]\frac{d\sigma_i}{dt}+\frac{i}{2(i+1)^2+1}\frac{gwa}{n\mathfrak{t}}\sigma_i=\frac{i(2i+1)}{2(i-1)[2(i+1)^2+1]}\frac{wa^{i+1}}{n}\left(\frac{1}{\mathfrak{t}}+\frac{d}{dt}\right)S_i.$$

This equation admits of solution just in the same way that equation (11) was solved; but I shall confine myself to the case of the tidal problem, where $i=2$ and $S_2=S\cos(vt+\eta)$. In this special case the equation becomes

$$\left(1+\frac{2gwa}{19n}\right)\frac{d\sigma}{dt}+\frac{2gwa}{19n\mathfrak{t}}\sigma=\frac{5wa^3}{19n}\left[\frac{1}{\mathfrak{t}}\cos(vt+\eta)-v\sin(vt+\eta)\right]S.$$

And if we put $\frac{19n}{2gwa}+1=\frac{1}{k}$, $\tan\psi=v\mathfrak{t}$, and $\mathfrak{g}=\frac{2g}{5a}$

This may be written

$$\frac{d\sigma}{dt}+\frac{k}{\mathfrak{t}}\sigma=\frac{vak}{\mathfrak{g}\sin\psi}S\cos(vt+\eta+\psi).$$

In the solution appropriate to the tidal problem, we may omit the exponential term, and assume $\sigma=A\cos(vt+B)$. Then if we put $\tan\chi=\frac{v\mathfrak{t}}{k}$

$$\frac{d\sigma}{dt}+\frac{k}{\mathfrak{t}}\sigma=\frac{Av}{\sin\chi}\cos(vt+B+\chi).$$

Whe it follows that $B=\eta+\psi-\chi$, and

$$A=\frac{a}{\mathfrak{g}}k\frac{\sin\chi}{\sin\psi}=\frac{a}{\mathfrak{g}}\frac{\cos\chi}{\cos\psi},$$

so that

$$\sigma=\frac{aS}{\mathfrak{g}}\frac{\cos\chi}{\cos\psi}\cos(vt+\eta+\psi-\chi).$$

Hence the bodily tide of the elastico-viscous spheroid is equal to the equilibrium tide of a fluid spheroid multiplied by $\frac{\cos\chi}{\cos\psi}$, and high tide is retarded by $\chi-\psi\div v$.

The formula for $\tan\chi$ may be expressed in a somewhat more convenient form; we have $\tan\psi=v\mathfrak{t}$, and therefore $\tan\chi=\tan\psi+\frac{19nv\mathfrak{t}}{2gwa}$

But $n\mathfrak{t}$ is the coefficient of viscosity, and in treating the tides of the purely viscous spheroid we put $\tan\epsilon=\frac{19v}{2gwa}\times$ coefficient of viscosity; therefore adopting the same notation here, we have $\tan\chi=\tan\psi+\tan\epsilon$.

If the modulus of relaxation $\mathfrak{t}$ be zero, whilst the coefficient of rigidity n becomes infinite, but $n\mathfrak{t}$ finite, the substance is purely viscous, and we have $\psi=0$ and $\chi=\epsilon$, so that the solution reduces to the case already considered. If $\mathfrak{t}$ be infinite, the substance is purely elastic, and we have $\psi=\frac{\pi}{2}, \chi=\frac{\pi}{2}$ and since $\frac{\cos\chi}{\cos\psi}=k\frac{\sin\chi}{\sin\psi}$, therefore

$$\sigma=\frac{ak}{\mathfrak{g}}\mathrm{S}\cos(vt+\eta).$$

But according to THOMSON's notation* $\frac{19n}{2gwa}=\frac{\mathfrak{r}}{\mathfrak{g}}$, so that $\sigma=\frac{a}{\mathfrak{r}+\mathfrak{g}}\mathrm{S}\cos(vt+\eta)$, which is the solution of THOMSON's problem of the purely elastic spheroid.

The present solution embraces, therefore, both the case considered by him, and that of the viscous spheroid.

9. *Ocean tides on an elastico-viscous nucleus.*

If $r=a+u$ be the equation to the ocean spheroid, we have, as in sec. (6), that the height of tide relatively to the nucleus is given by

$$u-\sigma=\frac{a^2}{g}\mathrm{S}\cos(vt+\eta)-\frac{2}{5}\sigma,$$

and substituting the present value of σ,

$$u-\sigma=\frac{2}{5}\frac{a}{\mathfrak{g}}\mathrm{S}\left[\cos(vt+\eta)-\frac{\cos\chi}{\cos\psi}\cos(vt+\eta+\psi-\chi)\right]$$
$$=-\frac{2}{5}\frac{a}{\mathfrak{g}}\mathrm{S}\frac{\sin(\chi-\psi)}{\cos\psi}\sin(vt+\eta-\chi).$$

If the nucleus had been rigid the rise and fall would have been given by $\mathrm{H}\cos(vt+\eta)$, where $\mathrm{H}=\frac{2}{5}\frac{a}{\mathfrak{g}}\mathrm{S}$; therefore on the yielding nucleus it is given by

$$u-\sigma=-\mathrm{H}\frac{\sin(\chi-\psi)}{\cos\psi}\sin(vt+\eta-\chi)$$
$$=-\mathrm{H}\cos\chi(\tan\chi-\tan\psi)\sin(vt+\eta-\chi)$$
$$=-\mathrm{H}\cos\chi\tan\epsilon\sin(vt+\eta-\chi).$$

* 'Nat. Phil.', § 840.

Hence the apparent tides on the yielding nucleus are equal to the corresponding tides on a rigid nucleus reduced in the proportion of $\cos\chi \tan\epsilon$ to unity, and there is an acceleration of the time of high water equal to $\frac{1}{v}\left(\frac{\pi}{2}-\chi\right)$.

As these analytical results present no clear meaning to the mind, I have compiled the following tables. I take the two cases considered by Sir W. THOMSON, where the spheroid has the rigidity of glass, and that of iron, and I work out the results for various times of relaxation of rigidity, for the semidiurnal and fortnightly tides. The last line in each division of each table is THOMSON'S result.

I may remind the reader that the modulus of relaxation of rigidity is the time in which the stress requisite to retain the body in its strained configuration falls to ·368 of its initial value.

SPHEROID with Rigidity of Glass ($2{\cdot}44\times10^8$).

Lunar Semidiurnal Tide.			
Modulus of relaxation of rigidity (t).	Coefficient of viscosity ($nt\times10^{-10}$).	Ocean tide is tide on rigid nucleus multiplied by ($\cos\chi\tan\epsilon$).	High tide relatively to nucleus is accelerated by $\left(\frac{\pi}{2}-\chi\right)\frac{1}{v}$
Hrs.			Hrs. min.
Fluid 0	0	·000	3 6
1	88	·256	1 44
2	176	·342	1 3
3	264	·370	0 45
4	351	·382	0 34
5	439	·388	0 28
Elastic ∞	∞	·398	0 0
Fortnightly Tide.			
Days. hrs.			Days. hrs.
Fluid 0 0	0	·000	3 10
0 6	500	·099	2 21
0 12	1,100	·181	2 9
1 0	2,100	·285	1 16
2 0	4,200	·357	1 0
3 0	6,300	·379	0 16
Elastic ∞	∞	·398	0 0

SPHEROID with Rigidity of Iron ($7 \cdot 8 \times 10^8$).

Lunar Semidiurnal Tide.

Modulus of relaxation.		Viscosity.	Reduction of ocean tide.	Acceleration of high water.	
Hrs.	min.			Hrs.	min.
Fluid 0	0	0	·000	3	6
0	30	140	·420	1	47
1	0	280	·573	1	7
2	0	560	·647	0	36
3	0	840	·665	0	25
Elastic	∞	∞	·679	0	0

Fortnightly Tide.

Modulus of relaxation.		Viscosity.	Reduction of ocean tide.	Acceleration of high water.	
Days.	hrs.			Days.	hrs.
Fluid 0	0	0	·000	3	10
0	6	1,700	·294	2	11
0	12	3,400	·470	1	18
1	0	6,700	·602	1	1
2	0	13,500	·657	0	13
3	0	20,200	·669	0	9
Elastic	∞	∞	·679	0	0

10. *The influence of inertia.*

In establishing these results inertia has been neglected, and I will now show that this neglect is not such as to materially vitiate my results.*

Suppose that the spheroid is constrained to execute such a vibration as it would do if it were a perfect fluid, and if the equilibrium theory of tides were true. Then the effective forces which are, according to D'ALEMBERT'S principle, the equivalent of inertia, are found by multiplying the acceleration of each particle by its mass.

Inertia may then be safely neglected if the effective force on that particle which has the greatest amplitude of vibration is small compared with the tide-generating force on it. In the case of a viscous spheroid, the inertia will have considerably less effect than it would have in the supposed constrained oscillation.

Now suppose we have a tide-generating potential $wr^2 \mathrm{S} \cos(vt+\eta)$, then, according to the equilibrium theory of tides, the form of the surface is given by

$$\sigma = \frac{5a^2}{2g} \mathrm{S} \cos(vt+\eta) ;$$

* In a future paper (read on December 19th, 1878) I shall give an approximate solution of the problem, inclusive of the effects of inertia.

and this function gives the proposed constrained oscillation. It is clear that it is the particles at the surface which have the widest amplitude of oscillation. The effective force on a unit element at the surface is

$$-w\frac{d^2\sigma}{dt^2}=\frac{5a^2}{2g}wv^2\,\mathrm{S}\cos(vt+\eta).$$

But the normal disturbing force at the surface is $2wa\,\mathrm{S}\cos(vt+\eta)$. Therefore inertia may be neglected if $\frac{5a^2}{2g}wv^2$ is small compared with $2wa$, or if $\frac{5a}{4g}v^2$ is a small fraction. The tide of the shortest period with which we have to deal is that in which $v=2\omega$, so that we must consider the magnitude of the fraction $4\times\frac{5a\omega^2}{4g}$. If ω were the earth's true angular velocity, instead of its angular velocity relatively to the moon, then $\frac{5a\omega^2}{4g}$ would be the ellipticity of its surface if it were homogeneous. This ellipticity is, as is well known, $\frac{1}{232}$. Hence the fraction, which is the criterion of the negligeability of inertia, is about $\frac{1}{58}$.

If, then, it be considered that this way of looking at the subject certainly exaggerates the influence of inertia, it is clear that the neglect of inertia is not such as to materially vitiate the results given above.

II.

A TIDAL YIELDING OF THE EARTH'S MASS, AND THE CANAL-THEORY OF TIDES.

In the first part of this paper the equilibrium theory has been used for the determination of the reduction of the height of tide, and the alteration of phase, due to bodily tides in the earth.

Sir W. THOMSON remarks, with reference to a supposed elastic yielding of the earth's body: "Imperfect as the comparisons between theory and observation as to the actual height of the tides has been hitherto, it is scarcely possible to believe that the height is in reality only two-fifths of what it would be if, as has been universally assumed in tidal theories, the earth were perfectly rigid. It seems, therefore, nearly certain, with no other evidence than is afforded by the tides, that the tidal effective rigidity of the earth must be greater than that of glass."*

The equilibrium theory is quite fallacious in its explanation of the semidiurnal tide, but Sir W. THOMSON is of opinion that it must give approximately correct results for tides of considerable period. It is therefore on the observed amount of the fortnightly tide that he places reliance in drawing the above conclusion. Under these

* 'Nat. Phil.', § 843.

circumstances, a dynamical investigation of the effects of a tidal yielding of the earth on a tide of short period, according to the canal theory, is likely to be interesting.

The following investigation will be applicable either to the case of the earth's mass yielding through elasticity, plasticity, or viscosity; it thus embraces Sir W. THOMSON'S hypothesis of elasticity, as well as mine of viscosity and elastico-viscosity.

11. *Semidiurnal tide in an equatorial canal on a yielding nucleus.*

I shall only consider the simple case of the moon moving uniformly in the equator, and raising tide waves in a narrow shallow equatorial canal of depth h.

The potential of the tide-generating force, as far as concerns the present inquiry, is, with the old notation, $\left(\frac{r}{a}\right)^2\frac{\tau}{2}\sin^2\theta\cos 2(\phi-\omega t)$, where $\tau=\frac{3}{2}\frac{ma^2}{c^3}$. This force will raise a bodily tide in the earth, whether it be elastic, plastic, or viscous. Suppose, then, that the greatest range of the bodily tide at the equator is 2E, and that it is retarded after the passage of the moon over the meridian by an angle $\frac{\epsilon}{2}$. Then the equation to the bounding surface of the solid earth, at the time t, is $r=a+\mathrm{E}\sin^2\theta\cos[2(\phi-\omega t)+\epsilon]$; or with former notation $\sigma=\mathrm{E}\sin^2\theta\cos[2(\phi-\omega t)+\epsilon]$.

The whole potential V, at a point outside the nucleus, is the sum of the potential of the earth's attraction, and of the potential of the tide-generating force. Therefore

$$\mathrm{V}=g\frac{a^2}{r}+\frac{3}{5}g\left(\frac{r}{a}\right)^2\mathrm{E}\sin^2\theta\cos[2(\phi-\omega t)+\epsilon]+\frac{\tau}{2}\left(\frac{r}{a}\right)^2\sin^2\theta\cos 2(\phi-\omega t)$$

$$=g\frac{a^2}{r}+\{\mathrm{F}\cos[2(\phi-\omega t)+\epsilon]+\mathrm{G}\sin[2(\phi-\omega t)+\epsilon]\}\left(\frac{r}{a}\right)^2\sin^2\theta$$

where $\mathrm{F}=\frac{3}{5}g\mathrm{E}+\frac{\tau}{2}\cos\epsilon$, $\mathrm{G}=\frac{\tau}{2}\sin\epsilon$.

Sir GEORGE AIRY shows, in his article on "Tides and Waves" in the 'Encyclopædia Metropolitana,' that the motion of the tide-wave in a canal running round the earth is the same as though the canal were straight, and the earth at rest, whilst the disturbing body rotates round it. This simplification will be applicable here also.

As before stated, the canal is supposed to be equatorial, and of depth h.

After the canal has been developed, take the origin of rectangular coordinates in the undisturbed surface of the water, and measure x along the canal in the direction of the moon's motion, and y vertically downwards.

We have now to transform the potential V, and the equation to the surface of the solid earth, so as to make them applicable to the supposed development. If v be the velocity of the tide-wave, then $\omega a=v$; also the wave length is half the circumference of the earth's equator, or πa; and let $m=\frac{2}{a}$. Then we have the following transformations:—

$$\theta=\frac{\pi}{2},\ \phi=\frac{mx}{2},\quad r=a+h-y.$$

Also in the small terms we may put $r=a$. Thus the potential becomes

$$\mathrm{V}=\text{const.}+gy+\mathrm{F}\cos\left[m(x-vt)+\epsilon\right]+\mathrm{G}\sin\left[m(x-vt)+\epsilon\right].$$

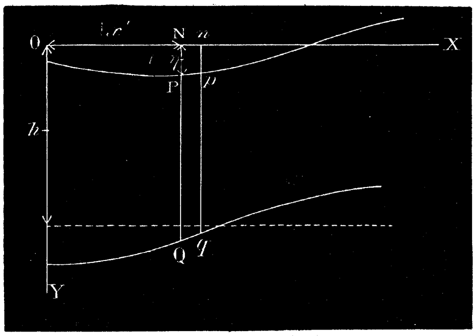
Again, to find the equation to the bottom of the canal, we have to transform the equation

$$r=a+\mathrm{E}\sin^2\theta\cos\left[2(\phi-\omega t)+\epsilon\right].$$

If y' be the ordinate of the bottom of the canal, corresponding to the abscissa x, this equation becomes after development

$$y'=h-\mathrm{E}\cos\left[m(x-vt)+\epsilon\right].$$

We now have to find the forced waves in a horizontal shallow canal, under the action of a potential V, whilst the bottom executes a simple harmonic motion. As the canal is shallow, the motion may be treated in the same way as Professor STOKES has treated the long waves in a shallow canal, of which the bottom is stationary. In this method it appears that the particles of water, which are at any time in a vertical column, remain so throughout the whole motion.



Suppose, then, that $x+\xi=x'$ is the abscissa of a vertical line of particles PQ, which, when undisturbed, had an abscissa x.

Let η be the ordinate of the surface corresponding to the abscissa x'.

Let pq be a neighbouring line of particles, which when undisturbed were distant from PQ a small length k.

Conceive a slice of water cut off by planes through PQ, pq perpendicular to the length of the canal, of which the breadth is b. Then the volume of this slice is $b\times\mathrm{PQ}\times\mathrm{N}n$.

$$\text{Now PQ}=h-\mathrm{E}\cos\left[m(x'-vt)+\epsilon\right]-\eta,$$

$$\text{and}\quad \mathrm{N}n=k\left(1+\frac{d\xi}{dx}\right).$$

Hence treating E and η as small compared with h, the volume of the slice is—

$$bhk\left\{1+\frac{d\xi}{dx}-\frac{E}{h}\cos\left[m(x'-vt)+\epsilon\right]-\frac{\eta}{h}\right\}.$$

But this same slice, in its undisturbed condition, had a volume bhk. Therefore the equation of continuity is

$$\eta=h\frac{d\xi}{dx}-E\cos\left[m(x'-vt)+\epsilon\right].$$

Now the hydrodynamical equation of motion is approximately

$$\frac{dp}{dx'}=\frac{dV}{dx'}-\frac{d^2\xi}{dt^2}$$

The difference of the pressures on the two sides of the slice $PQqp$ at any depth is $Nn\times\frac{dp}{dx'}$; and this only depends on the difference of the depressions of the wave-surface below the axis of x on the two sides of the slice, viz. at P and p. Thus $\frac{dp}{dx'}=-g\frac{d\eta}{dx'}$

Substituting then for η from the equation of continuity, and observing that $\frac{d^2\xi}{dxdx'}$ is very nearly the same as $\frac{d^2\xi}{dx^2}$, we have as the equation of wave motion,

$$gh\frac{d^2\xi}{dx^2}+mg\,E\sin\left[m(x'-vt)+\epsilon\right]=-\frac{dV}{dx'}+\frac{d^2\xi}{dt^2}$$

But

$$\frac{dV}{dx'}=-m\,F\sin\left[m(x'-vt)+\epsilon\right]+m\,G\cos\left[m(x'-vt)+\epsilon\right].$$

So that

$$\frac{d^2\xi}{dt^2}=gh\frac{d^2\xi}{dx^2}+m\{G\cos\left[m(x'-vt)+\epsilon\right]-(F-Eg)\sin\left[m(x'-vt)+\epsilon\right]\}.$$

In obtaining the integral of this equation, we may omit the terms which are independent of G, F, E, because they only indicate free waves, which may be supposed not to exist.

The approximation will also be sufficiently close, if x be written for x' on the right hand side.

Assume, then, that

$$\xi=A\cos\left[m(x-vt)+\epsilon\right]+B\sin\left[m(x-vt)+\epsilon\right].$$

By substitution in the equation of motion we find

$$-m^2(v^2-gh)\{A\cos+B\sin\}=m\{G\cos-(F-Eg)\sin\}.$$

And as this must hold for all times and places,

$$A=-\frac{G}{m(v^2-gh)}=\frac{-\frac{1}{2}a\tau\sin\epsilon}{2(a^2\omega^2-gh)}$$

$$B=\frac{F-Eg}{m(v^2-gh)}=\frac{a(\frac{\tau}{2}\cos\epsilon-\frac{2}{5}gE)}{2(a^2\omega^2-gh)}$$

In the case of such seas as exist in the earth, the tide-wave travels faster than the free-wave, so that $a^2\omega^2$ is greater than gh; and the denominators of A and B are positive.

We have then—

$$\xi=\frac{a}{2(a^2\omega^2-gh)}\left\{\left(\frac{\tau}{2}\cos\epsilon-\frac{2}{5}gE\right)\sin-\frac{\tau}{2}\sin\epsilon\cos\right\}$$

But the present object is to find the motion of the wave-surface relatively to the bottom of the canal, for this will give the tide relatively to the dry land. Now the height of the wave relatively to the bottom is

$$PQ=h-E\cos[m(x-vt)+\epsilon]-\eta$$
$$=h-h\frac{d\xi}{dx}$$

And

$$\frac{d\xi}{dx}=\frac{1}{a^2\omega^2-gh}\left\{\left(\frac{\tau}{2}\cos\epsilon-\frac{2}{5}gE\right)\cos+\frac{\tau}{2}\sin\epsilon\sin\right\}$$

Hence reverting to the sphere, and putting a for $a+h$, we get as the equation to the relative spheroid of which the wave-surface in the equatorial canal forms part—

$$r=a-\frac{h\sin^2\theta}{a^2\omega^2-gh}\left\{\frac{\tau}{2}\cos 2(\phi-\omega t)-\frac{2}{5}gE\cos[2(\phi-\omega t)+\epsilon]\right\}$$

But according to the equilibrium theory, if V has the same form as above, viz.—

$$g\frac{a^2}{r}+\frac{3}{5}g\left(\frac{r}{a}\right)^2 E\sin^2\theta\cos[2(\phi-\omega t)+\epsilon]+\frac{\tau}{2}\left(\frac{r}{a}\right)^2\sin^2\theta\cos 2(\phi-\omega t)$$

and if $r=a+u$ be the equation to the tidal spheroid, we have, as in Part I.,

$$u=\frac{\sin^2\theta}{g}\left\{\frac{\tau}{2}\cos 2(\phi-\omega t)+\frac{3}{5}gE\cos[2(\phi-\omega t)+\epsilon]\right]$$

and the equation to the relative tidal spheroid is

$$r=a+u-\sigma$$
$$=a+\frac{\sin^2\theta}{g}\left\{\frac{\tau}{2}\cos 2(\phi-\omega t)-\frac{2}{5}gE\cos[2(\phi-\omega t)+\epsilon]\right\}$$

Now in either the case of the dynamical theory or of the equilibrium theory, if E be put equal to zero, we get the equations to the tidal spheroid on a rigid nucleus. A comparison, then, of the above equations shows at once that both the reduction of tide and the acceleration of phase are the same in one theory as in the other. But where the one gives high water, the other gives low water. The result is applicable to any kind of supposed yielding of the earth's mass; and in the special case of viscosity, the table of results for the fortnightly tide at the end of Part I. is applicable.

III.

SUMMARY AND CONCLUSIONS.

In § 1 an analogy is shown between problems about the state of strain of incompressible elastic solids, and the flow of incompressible viscous fluids, when inertia is neglected; so that the solutions of the one class of problems may be made applicable to the other. Sir W. THOMSON'S problem of the bodily tides of an elastic sphere is then adapted so as to give the bodily tides of a viscous spheroid. The adaptation is rendered somewhat complex by the necessity of introducing the effects of the mutual gravitation of the parts of the spheroid.

The solution is only applicable where the disturbing potential is capable of expansion as a series of solid harmonics, and it appears that each harmonic term in the potential then acts as though all the others did not exist; in consequence of this it is only necessary to consider a typical term in the potential.

In § 3 an equation is found which gives the form of the free surface of the spheroid at any time, under the action of any disturbing potential, which satisfies the condition of expansibility. By putting the disturbing potential equal to zero, the law is found which governs the subsidence of inequalities on the surface of the spheroid, under the influence of mutual gravitation alone. If the form of the surface be expressed as a series of surface harmonics, it appears that any harmonic diminishes in geometrical progression as the time increases in arithmetical progression, and harmonics of higher orders subside much more slowly than those of lower orders. Common sense, indeed, would tell us that wide-spread inequalities must subside much more quickly than wrinkles, but only analysis could give the law connecting the rapidity of the subsidence with the magnitude of the inequality.*

* On this Lord RAYLEIGH remarks, that if we consider the problem in two dimensions, and imagine a number of parallel ridges, the distance between which is λ, then inertia being neglected, the elements on which the time of subsidence depends are gw (force per unit mass due to weight), v the coefficient of viscosity, and λ. Thus the time T must have the form

$$T=(gw)^x v^y \lambda^z.$$

The dimensions of gw, v, λ are respectively $ML^{-2}T^{-2}$, $ML^{-1}T^{-1}$, L; hence

$$x+y=0$$
$$-2x-y+z=0$$
$$-2x-y=1,$$

$$\text{And } x=-1,\ y=1,\ z=-1, \text{ so that T varies as } \frac{v}{gw\lambda}.$$

If we take the case on the sphere, then when i, the order of harmonics, is great, λ compares with $\frac{a}{i}$; so that T varies as $\frac{vi}{gwa}$.

I hope at some future time to try whether it will not be possible to throw some light on the formation of parallel mountain chains and the direction of faults, by means of this equation. Probably the best way of doing this will be to transform the surface harmonics, which occur here, into BESSEL'S functions.

In § 4 the rate is considered at which a spheroid would adjust itself to a new form of equilibrium, when its axis of rotation had separated from that of figure; and the law is established which was assumed in a previous paper.*

In § 5 I pass to the case where the disturbing potential is a solid harmonic of the second degree, multiplied by a simple time harmonic. This is the case to be considered for the problem of a tidally distorted spheroid. A remarkably simple law is found connecting the viscosity, the height of tide, and the amount of lagging of tide; it is shown that if v be the speed of the tide, and if $\tan \epsilon$ varies jointly as the coefficient of viscosity and v, then the height of bodily tide is equal to that of the equilibrium tide of a perfectly fluid spheroid multiplied by $\cos \epsilon$, and the tide lags by a time equal to $\frac{\epsilon}{v}$.

It is then shown (§ 6) that in the equilibrium theory the *ocean* tides on the yielding nucleus will be equal in height to the ocean tides on a rigid nucleus multiplied by $\sin \epsilon$, and that there will be an acceleration of the time of high water equal to $\frac{\pi}{2v}-\frac{\epsilon}{v}$.

The tables in § 7 give the results of the application of the preceding theories to the lunar semidiurnal and fortnightly tides for various degrees of viscosity. A comparison of the numbers in the first columns with the viscosity of pitch at near the freezing temperature (viz., about $1\cdot 3\times 10^8$, as found by me), when it is hard, apparently solid and brittle, shows how enormously stiff the earth must be to resist the tidally deforming influence of the moon. For unless the viscosity were very much larger than that of pitch, the viscous sphere would comport itself sensibly like a perfect fluid, and the ocean tides would be quite insignificant. It follows, therefore, that no very considerable portion of the interior of the earth can even distantly approach the fluid state.

This does not, however, seem to be conclusive against the existence of bodily tides in the earth of the kind here considered; for although (as remarked by Sir W. THOMSON) a very great hydrostatic pressure probably has a tendency to impart rigidity to a substance, yet the very high temperature which must exist in the earth at a small depth would tend to induce a sort of viscosity—at least if we judge by the behaviour of materials at the earth's surface.

In § 8 the theory of the tides of an imperfectly elastic spheroid is developed. The kind of imperfection of elasticity considered is where the forces requisite to maintain the body in any strained configuration diminish in geometrical progression as the time increases in arithmetical progression. There can be no doubt that all bodies *do* possess an imperfection in their elasticity of this general nature, but the exact law

* Phil. Trans., Vol. 167, Part I., sec. 5 of my paper.

here assumed has not, as far as I am aware, any experimental justification; its adoption was rather due to mathematical necessities than to any other reason.

It would, of course, have been much more interesting if it had been possible to represent more exactly the mechanical properties of solid matter. One of the most important of these is that form of resistance to relative displacement, to which the term "plasticity" has been specially appropriated. This form of resistance is such that there is a change in the law of resistance to the relative motion of the parts, when the forces tending to cause flow have reached a certain definite intensity. This idea was founded, I believe, by MM. TRESCA and ST. VÉNANT on a long course of experiments on the punching and squeezing of metals;* and they speak of a solid being reduced to the state of fluidity by stresses of a given magnitude. This theory introduces a discontinuity, since it has to be determined what parts of the body are reduced to the state of fluidity and what are not. But apart from this difficulty, there is another one which is almost insuperable, in the fact that the differential equations of flow are non-linear.

The hope of introducing this form of resistance must be abandoned, and the investigation must be confined to the inclusion of those two other continuous laws of resistance to relative displacement—elasticity and viscosity.

As above stated, the law of elastico-viscosity assumed in this paper has not got an experimental foundation. Indeed, KOHLRAUSCH's experiments on glass† show that the elasticity degrades rapidly at first, and that it tends to attain a final condition, from which it does not seem to vary for an almost indefinite time. But glass is one of the most perfectly elastic substances known, and, by the light of TRESCA's experiments, it seems probable that experiments with lead would have brought out very different results. It seems, moreover, hardly reasonable to suppose that the materials of the earth possess much mechanical similarity with glass. Notwithstanding all these objections, I think, for my part, that the results of this investigation of the tides of an ideal elastico-viscous sphere are worthy of attention.

There are two constants which determine the nature of this ideal solid: first, the coefficient of rigidity, at the instant immediately after the body has been placed in its strained configuration; and secondly, "the modulus of the time of relaxation of rigidity," which is the time in which the force requisite to retain the body in its strained configuration has fallen away to ·368 of its initial value.

In this section it is shown that the equations of flow of this incompressible elastico-viscous body have the same mathematical form as those for a purely viscous body; so that the solutions already attained are easily adapted to the new hypothesis.

The only case where the problem is completely worked out, is when the disturbing

* "Sur l'écoulement des Corps Solides," Mém. des Savants Étrangers, tom. xviii. and tom. xx., p. 75 and p. 137. See also 'Comptes Rendus,' tom. 66, 68, and Liouville's Journ., 2[me] série, xiii., p. 379, and xvi., p. 308, for papers on this subject.

† POGGENDORFF Ann., vol. 119, p. 337.

potential has the form appropriate to the tidal problem. The laws of reduction of bodily tide, of its lagging, of the reduction of ocean tide, and of its acceleration, are somewhat more complex than in the case of pure viscosity; and the reader is referred to § 8 for the statement of those laws. It is also shown that by appropriate choice of the values of the two constants, the solutions may be either made to give the results of the problem for a purely viscous sphere, or for a purely elastic one.

The tables give the results, for the semidiurnal and fortnightly tides, of this theory for spheroids which have the rigidity of glass or of iron—the two cases considered by Sir W. THOMSON. As it is only possible to judge of the amount of bodily tide by the reduction of the ocean tide, I have not given the heights and retardations of the bodily tide.

It appears that if the time of relaxation of rigidity is about one quarter of the tidal period, then the reduction of ocean tide does not differ much from what it would be if the spheroid were perfectly elastic. The amount of tidal acceleration still, however, remains considerable. A like observation may be made with respect to the acceleration of tide in the case of pure viscosity approaching rigidity: and this leads me to think that one of the most promising ways of detecting such tides in the earth would be by the determination of the periods of maximum and minimum in a tide of long period, such as the fortnightly in a high latitude.

In § 10 it is shown that the effects of inertia, which had been neglected in finding the laws of the tidal movements, cannot be such as to materially affect the accuracy of the results.

[* The hypothesis of a viscous or imperfectly elastic nature for the matter of the earth would be rendered extremely improbable, if the ellipticity of an equatorial section of the earth were not very small. An ellipsoidal figure with three unequal axes, even if theoretically one of equilibrium, could not continue to subsist very long, because it is a form of greater potential energy than the oblate spheroidal form, which is also a figure of equilibrium.

Now, according to the results of geodesy, which until very recently have been generally accepted as the most accurate—namely, those of Colonel A. R. CLARKE†—there is a difference of 6,378 feet between the major and minor equatorial radii, and the meridian of the major axis is 15° 34′ E. of Greenwich.

The heterogeneity of the earth would have to be very great to permit so large a deviation from the oblate spheroidal shape to be either permanent, or to subside with extreme slowness. But since this paper was read, Colonel CLARKE has published a revision of his results, founded on new data;‡ and he now finds the difference between the equatorial radii to be only 1,524 feet, whilst the meridian of the greatest axis is 8° 15′ west. This exhibits a change of meridian of 24°, and a reduction of equatorial

* The part within brackets [ ] was added in November, 1878, in consequence of a conversation with Sir W. THOMSON.

† Quoted in THOMSON and TAIT, Nat. Phil., sec. 797.

‡ Phil. Mag., August, 1878.

ellipticity to about one quarter of the formerly-received value. Moreover, the new value of the polar axis is about 1,000 feet larger than the old one.

Colonel CLARKE himself obviously regards the ellipsoidal form of the equator as doubtful. Thus there is at all events no proved result of geodesy opposed to the present hypothesis concerning the constitution of the earth. Sir W. THOMSON remarks in a letter to me that "we may look to further geodetic observations and revisals of such calculations as those of Colonel CLARKE for verification or disproof of your viscous theory."]

In the first part of the paper the equilibrium theory is used in discussing the question of ocean tides; in the second part I consider what would be the tides in a shallow equatorial canal running round the equator, if the nucleus yielded tidally at the same time. The reasons for undertaking this investigation are given at the beginning of that part. In § 11 it is shown that the height of tide relatively to the nucleus bears the same proportion to the height of tide on a rigid nucleus as in the equilibrium theory, and the alteration of phase is also the same; but where the one theory gives high water the other gives low water.

The chief practical result of this paper may be summed up by saying that it is strongly confirmatory of the view that the earth has a very great effective rigidity. But its chief value is that it forms a necessary first chapter to the investigation of the precession of imperfectly elastic spheroids, which will be considered in a future paper.* I shall there, as I believe, be able to show, by an entirely different argument, that the bodily tides in the earth are probably exceedingly small at the present time.

APPENDIX.

November 7, 1878.

On the observed height and phase of the fortnightly oceanic tide.

In the following note I attempt to carry out the suggestion concerning the fortnightly tide made in the preceding paper.

The reports of the Tidal Committee of the British Association for 1872 and 1876 contain the reductions of the tidal observations at a number of stations, into a series of harmonic tides, corresponding to the theoretical harmonic constituents of the tide-generating forces of the moon and sun. The tide with which we are here concerned is the fortnightly declinational tide.

The heights of the tides at various times are all expressed in the form R cos $(nt-\epsilon)$, where R is half the range of the tide in English feet, n the "speed" of the tide, and ϵ the retardation of phase, so that $\epsilon \div n$ is the "lag" of the tide.

* Read before the Royal Society on December 19th, 1878.

With the notation of the present paper $n=2\Omega$ for the fortnightly tide, and Ωt is the "mean moon's" longitude from her node.

The following are the results, giving the place of observation, its N. latitude, and the years of observations. With respect to Brest and Toulon, R is reduced to feet from centimètres, so as to be made comparable with the other results :—

Ramsgate, about 51° 21′.	Liverpool, 53° 40′.				Hartlepool, 54° 41′.		
1864.	1857–58.	1858-59.	1859-60.	1866-67.	1858-59.	1859-60.	1860-61.
R ·0331 ϵ 268°·29	·093 170°·7	·037 148°·8	·024 72°·9	·036 340°·6	·052 190°·34	·053 222°·34	·073 158°·62

Brest, 48° 23′.	Toulon, 43° 7′.	Kurrachee, 24° 53′.			Cat Island, Gulf of Mexico, 30° 23′.
1875.	1853.	1868–69.	1869-70.	1870 71.	—
R ·099 ϵ 80°·65	·051 139°·50	·038 335°·40	·064 333°·91	·035 283°·22	·043 136°·69

In their present form the observations do not appear to present any semblance of law, but when they are rearranged we shall be able to form some idea as to whether they are really quite valueless or not for the point under consideration.

The theoretical expression for the fortnightly tide of an ocean covering the whole earth, according to the equilibrium theory, is

$$\frac{3}{10}\frac{\tau}{\mathfrak{g}} a \sin^2 i(\tfrac{1}{3}-\cos^2\theta) \cos 2\Omega t.$$

Where $\tau=\frac{3m}{2c^3}$, $\mathfrak{g}=\frac{2g}{5a}$, $a=$ earth's radius, i the average obliquity of the earth's axis to the normal to the plane of the lunar orbit during the fortnight in question, θ the colatitude of the place of observation.

If we take $i=23° 28'$ the obliquity of the ecliptic, $a=20{\cdot}9$ million feet, we find

$$\frac{3}{10}\frac{\tau}{\mathfrak{g}} a \sin^2 i = {\cdot}207 \text{ foot.}$$

So that the fortnightly tide should be expressible by

$${\cdot}207\ (\tfrac{1}{3}-\sin^2(\text{lat.})) \cos 2\Omega t.$$

In THOMSON's corrected equilibrium theory the second factor should be

$$\frac{1+\mathfrak{C}}{3}-\sin^2(\text{lat.})$$

where $\mathfrak{C}$ is a certain definite integral, depending on the distribution of land and water, but which has not yet been evaluated.

The latitude of evanescent fortnightly tide is 36° 15′ if $\mathfrak{C}$ is zero; and if we bear in mind that $\mathfrak{C}$ may be negative, it is clear that the observations at Cat Island (lat. 30° 23′) are made too near the critical latitude to be trustworthy for determining the true fortnightly tide. It is also hardly possible to believe that the observations at Toulon should show a true tide of this denomination, because the Mediterranean must be regarded as a virtually closed sea. The observations at Cat Island, and at Toulon, will therefore be set aside.

The first process to be applied to the above observations is obviously to divide each value of R by $\frac{1}{3}-\sin^2$ (lat.); the following are the factors for reducing the values of R:—

$[\frac{1}{3}-\sin^2(\text{lat.})]^{-1}$

Ramsgate.	3·62
Liverpool.	3·17
Hartlepool.	3·01
Brest.	3·07
Kurrachee.	6·40

These factors will be applied to the values of R in the table first given.

The next point to consider is the phase of the tide. The formula we have given shows that the fortnightly tide consists in an alternate deformation of the ocean level into an oblate and prolate spheroid of revolution, when the tide is deemed to be superposed on a true sphere, instead of on an oblate nucleus.

When t is zero the spheroid is oblate, and this may be called high-tide; when $t=\frac{\pi}{2\Omega}$ it may be called low-tide. It follows, therefore, that N. of lat. 36° 15′ high-*tide* is low-*water*, and *vice-versâ*; but S. of this latitude the tide and water agree. But the formulæ in the tidal reductions always refer to high-*water*, hence to find the retardation of the *tide* we must subtract 180° from all the ϵ's for places N. of 36° 15′—that is to say (Cat Island being rejected) for all except Kurrachee.

For Kurrachee, we may observe that any retardation ϵ may be regarded as a retardation $\epsilon-2\pi$, which, if negative, is an acceleration of tide. If $2\pi-\epsilon$ be less than 180°, this appears to be the more correct light in which to look at it.

Now if we reduce all the observations in the way indicated, so that the fortnightly tide is given by $R'(\frac{1}{3}-\cos^2\theta)\cos(2\Omega t-\eta)$, we find the following results :—

	Ramsgate.	Liverpool.				Hartlepool.		
R'	·120	·295	·117	·076	·114	·157	·160	·220
η	$+88°·29$	$-9°·3$	$-31°·2$	$-107°·1$	$+160°·6$	$+10°·34$	$+42°·34$	$-21°·38$

	Brest.	Kurrachee.		
R'	·304	·243	·410	·224
η	$-99°·35$	$-24°·60$	$-26°·09$	$-76°·78$

We will consider R' first.

From these twelve values we find $R'=·203$, with a probable error $\pm·068$.

The value of R' is almost exactly that indicated by theory (viz., ·207), but the very large probable error renders the result so uncertain, that it can only be asserted that the results do not disprove a diminution of fortnightly tide.

With regard to phase, it will be observed that there are eight cases of accelerated tide to four of retarded. Two of the retarded tides refer to Hartlepool, and concerning this station Sir W. THOMSON says in the report: "There is scarcely sufficient agreement between the results deduced from the long-period tides to be satisfactory, although the quantities of some are within reasonable limits."

It may be remarked, in passing, that Cat Island gives a retarded tide, and Toulon an accelerated one.

If we treat these alterations of phase in the same way as R' was treated, we find a mean acceleration of phase of $7°·85$, but with a *p.e.* several times larger than the result itself. But, in fact, with so few and such irregular observations the method of least squares is useless.

The cases of retarded phase certainly show considerably more irregularity than those of accelerated phase. If we take the mean of the cases with accelerated phase, we shall find an acceleration of 48°, which corresponds in time to an acceleration of 1 day 20 hours.

Now three out of four years of observations show an accelerated tide at Liverpool; all three years show an acceleration at Kurrachee; the Hartlepool observations are not of very much value; while the single year of acceleration at Brest may be set off against the single year of retardation at Ramsgate. If then we ask ourselves whether acceleration or retardation is the more probable, I think it must be answered in favour of acceleration; and if so there seem to be some indications of a viscous yielding of the earth's mass. It must be admitted, however, that the evidence is exceedingly uncertain.

It does not seem to be noticed in the tidal reports, that amongst the "Helmholtz compound shallow-water tides" there will be found several which have the same period, or very nearly the same period, as the true fortnightly declinational tide. If we write (as in the report) γ, σ for earth's rotaton and moon's mean motion, then we shall find the following speeds will combine so as to give shallow-water tides indistinguishable from the true fortnightly tide, namely, $2(\gamma-\sigma)$ and 2γ, also $\gamma-2\sigma$ and γ; and besides there are four combinations of the elliptic tides which give the same period of compound tide, if we neglect the motion of the moon's perigee. It therefore seems quite possible that in certain stations the true fortnightly tide may be masked, or have its phase largely affected by these compound tides, and this is, perhaps, the explanation of the great irregularity in the phases.

A series of observations in some oceanic island near the Equator, or better still far north or south, would be of immense value to decide this point.